2^{nd} Order ODE: Cauchy-Euler Equation

Hishammudin Afifi Bin Huspi
Faculty of Engineering
Universiti Malaysia Sarawak

Second-Order Differential Equations

Learning Objective:

1. To Cauchy-Euler Equation to Non Homogeneous 2nd order ODE
2. To identify the types of 2nd order Ordinary Differential Equation

Cauchy-Euler Equation

Cauchy-Euler Equation

- Applicable to solve linear equation with variable coefficients whose general solution can always be expressed in terms or powers of x, sines, cosines and logarithmic functions.

- Method of solution is quite similar to that for constant-coefficients equations in that an auxiliary equation must be solved.

Cauchy-Euler Equation

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots\dots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where the coefficients $a_n, a_{n-1,\dots\dots,} a_0$ are constants, is known as Cauchy-Euler equation.

The characteristics of this type of equation is that:

same　　　　　　same

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}}$$

Cauchy-Euler Equation

The homogeneous second order equation will be:

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0$$

Similar to constant coefficients, when we substitute $y=x^m$:

$$\begin{aligned}ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy &= am(m-1)x^m+bmx^m+cx^m\\ &=(am(m-1)+bm+c)x^m\end{aligned}$$

Thus, the auxiliary equation is $am^2+(b-a)m+c=0$

Cauchy-Euler Equation

And the nonhomogeneous equation

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=g(x)$$

Can be found once we have determined the complementary function y_c, by variation of parameters.

Cauchy-Euler Equation

There are 3 different cases to be considered, almost similar to constant coefficients method:

Case I: Distinct Real Roots $y = c_1 x^{m_1} + c_2 x^{m_2}$

Case II: Repeated Real Roots $y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

Case III: Conjugate Complex Roots

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Cauchy-Euler Equation

Example 1:

Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$

Solution:

Write in auxiliary equation $am^2 + (b-a)m + c = 0$

$$m^2 + (-2-1)m - 4 = 0$$

$$m^2 - 3m - 4 = 0$$

$$(m+1)(m-4) = 0 \quad \text{(Case I)}$$

For Case I, the general solution is $y = c_1x^{m_1} + c_2x^{m_2}$

$$y = c_1x^{-1} + c_2x^4$$

Cauchy-Euler Equation

Example 2:

Solve $4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$

Solution:

Write in auxiliary equation $am^2 + (b-a)m + c = 0$

$$4m^2 + (8-4)m + 1 = 0$$

$$4m^2 + 4m + 1 = 0$$

$$(2m+1)^2 = 0 \qquad \text{(Case II)}$$

For Case II, the general solution is $y = c_1x^{m_1} + c_2x^{m_1}\ln x$

$$y = c_1x^{-1/2} + c_2x^{-1/2}\ln x$$

Cauchy-Euler Equation

Example 3:

Solve $4x^2 y''+17y=0$

Solution:

Write in auxiliary equation $am^2+(b-a)m+c=0$

$$4m^2-4m+17=0$$

$$m_1=\frac{1}{2}+2i \quad and \quad m_2=\frac{1}{2}-2i$$ (Case III)

For Case III, the general solution is $y=x^{\alpha}[c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)]$

$$y=x^{1/2}\left[c_1\cos(2\ln x)+c_2\sin(2\ln x)\right]$$

The End