2nd Order ODE: Variation of Parameters

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Second-Order Differential Equations

Learning Objective:

1. To apply Variation of Parameters to Non Homogeneous 2nd order ODE
2. To identify the types of 2nd order Ordinary Differential Equation

Variation of Parameters

Variation of Parameters

- Variation of parameters has a distinct advantage over the method of undetermined coefficients because it will always produce a particular solution yp, provided that the associated homogeneous equation can be solved.

- It is not limited to a function f(x) that is a combination of four types as shown in undetermined coefficients method.

Variation of Parameters

A linear second-order differential equation

$$a_2(x)y''+a_1(x)y'+a_0(x)y = g(x) \qquad (1)$$

Can be written into the standard form,

$$y''+p(x)y'+q(x)y = f(x) \qquad (2)$$

Variation of Parameters

To solve equation (1):

1. Find the complementary function $y_c = c_1 y_1 + c_2 y_2$
2. Calculate Wronskian $W(y_1, y_2)$
3. Put to standard form (2) to determine f(x).
4. Find $u_1' = W_1 / W$ and $u_2' = W_2 / W$

   where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

5. Integrates u_1' and u_2' to find u_1 and u_2.
6. Form the particular solution $y_p = u_1 y_1 + u_2 y_2$
7. Finally, form the general solution $y = y_c + y_p$

Variation of Parameters

Example 1:

Find general solution for $y''-4y'+4y=(x+1)e^{2x}$

Solution:

1. *Find the complementary function* $y_c = c_1y_1 + c_2y_2$
   Homogeneous part, $y''-4y'+4y=0$
   Auxiliary equation, $m^2-4m+4=0$

$$(m-2)(m-2)$$

$$m_1 = m_2 = 2$$

Thus, $y_c = c_1e^{2x} + c_2xe^{2x}$

Variation of Parameters

Solution (continue...):

2. *Find Wronskian:*

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

3. *Put DE in standard form* $y''+p(x)y'+q(x)y = f(x)$

DE $y''-4y'+4y = (x+1)e^{2x}$ *is already in standard form.*

So, $f(x) = (x+1)e^{2x}$

Variation of Parameters

Solution (continue...):

4. *Find* $u_1' = W_1 / W$ *and* $u_2' = W_2 / W$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix}$$

$$= (x+1)e^{4x}$$

$$u_1' = \frac{W_1}{W} = \frac{-(x+1)xe^{4x}}{e^{4x}}$$

$$= -x(x+1)$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}}$$

$$= (x+1)$$

Variation of Parameters

Solution (continue…):

5. *Find u_1 and u_2:*

$$u_1' = -x(x+1)$$

$$u_1 = -\int x(x+1)dx$$

$$= -\left[\frac{x^3}{3} + \frac{x^2}{2}\right]$$

$$u_2' = (x+1)$$

$$u_2 = \int (x+1)dx$$

$$= \frac{x^2}{2} + x$$

Variation of Parameters

Solution (continue…):

6. *Particular solution is* $y_p = u_1 y_1 + u_2 y_2$:

$$y_1 = e^{2x} \quad and \quad y_2 = xe^{2x} \qquad u_1 = -\frac{x^3}{3} - \frac{x^2}{2} \quad and \quad u_2 = \frac{x^2}{2} + x$$

Thus, particular solution $y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$

$$= \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

7. *Finally, general solution,* $y = y_c + y_p$:

$$y = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

Appendix-Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$