2nd Order ODE: Undertemined Coefficients

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Second-Order Differential Equations

Learning Objective:

1. To apply Undetermined Coefficients to Non Homogeneous 2nd order ODE
2. To identify the types of 2nd order Ordinary Differential Equation

Second-Order Differential Equations

Chapter Contents:

- **Preliminary Theory on Linear Equations**
- **Reduction of Order**
- **Homogeneous Linear Equations with Constant Coefficients**
- Undetermined Coefficients
- Variation of Parameters
- Cauchy-Euler Equation

Second-Order Differential Equations

- Reduction of Order
- Homogeneous Linear Equations with Constant Coefficients

To solve Homogeneous Second Order DE

- Undetermined Coefficients
- Variation of Parameters
- Cauchy-Euler Equation

To solve Nonhomogeneous Second Order DE

Undetermined Coefficients

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

To solve a nonhomogeneous linear differential equation,

$$y''+p(x)y'+q(x)y=g(x)$$

we must do 2 things:

1. Find the complementary function, $y_c = c_1 y_1 + c_2 y_2$
2. Find the particular solution y_p of the nonhomogeneous equation.

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

Thus, we can obtain the general solution:

$$y = y_c + y_p$$

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

The disadvantage of this method is that it is limited to linear differential equations, such as:

- the coefficients are constant, and
- g(x) is a:
  1. Constant k,
  2. Polynomial function
  3. Exponential function $(e^{\alpha x})$
  4. Sine or cosine function

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

Some examples of functions that are appropriate for this method:

$$g(x) = 10$$

$$g(x) = \sin 3x - 5x \cos 2x$$

$$g(x) = x^2 - 5x$$

$$g(x) = 15x - 6 + 8e^{-x}$$

$$g(x) = xe^x \sin x + (3x^2 - 1)e^{-4x}$$

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

Some examples of functions that are **NOT** applicable for this method:

$$g(x) = \ln x$$

$$g(x) = \frac{1}{x}$$

$$g(x) = \tan x$$

$$g(x) = \sin^{-1} x$$

Undetermined Coefficients

Example 1:

Find the general solution for $y''+4y'-2y=2x^2-3x+6$

Solution:

Step 1 *Solve the associated homogeneous equation* $y''+4y'-2y=0$

Find the roots of the auxiliary equation $m^2+4m-2=0$.

Using formula, $m=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ *found that,* $m_1=-2-\sqrt{6}$ *and* $m_2=-2+\sqrt{6}$

Thus, complementary function is $y_c=c_1e^{\left(-2-\sqrt{6}\right)x}+c_2e^{\left(-2+\sqrt{6}\right)x}$.

Step 2 *Because g(x) is a quadratic polynomial, let's assume a particular solution that is also in the form of a quadratic polynomial:* $y_p=Ax^2+Bx+C$

We need to find the specific coefficients A, B and C for which y_p *is a solution of the differential equation* $y''+4y'-2y=2x^2-3x+6$.

Undetermined Coefficients

Solution (continue…):

Step 2 (cont.)

Differentiate $y_p = Ax^2 + Bx + C$ *twice:*

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

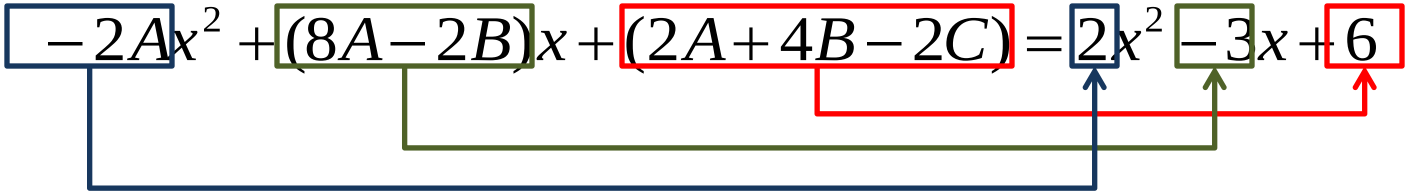
Substitute y_p and its derivative into differential equation:

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

Rearrange: $-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$

From the last equation we observe that:



$$\boxed{-2A}x^2 + \boxed{(8A - 2B)}x + \boxed{(2A + 4B - 2C)} = \boxed{2}x^2 \boxed{-3}x + \boxed{6}$$

Undetermined Coefficients

Solution (continue...):

Step 2 (cont.)

So, $-2A = A$, $8A - 2B = -3$, $2A + 4B - 2C = 6$

Solve $A = -1$, $8(-1) - 2B = -3$, $2(-1) + 4(-5/2) - 2C = 6$

$B = -5/2$

$-2 - 10 - 2C = 6$

$C = -9$

Thus, from $$y_p = Ax^2 + Bx + C$$

The particular solution is $$y_p = -x^2 - \frac{5}{2}x - 9$$

Step 3 *The general solution of the given differential solution is*

$$y = y_c + y_p$$

$$y = c_1 e^{\left(-2-\sqrt{6}\right)x} + c_2 e^{\left(-2+\sqrt{6}\right)x} - x^2 - \frac{5}{2}x - 9$$

Undetermined Coefficients

Example 2:

Find the particular solution for $y''-y'+y=2\sin 3x$

Solution:

To find particular solution means we are only interested in y_p.

Because twice differentiation of sin 3x produce 3cos 3x and -9sin 3x, we assume that the particular solution is

$$y_p = A\cos 3x + B\sin 3x$$

Differentiate y_p *twice gives:*

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

Substitute into DE:

$$y''-y'+y=2\sin 3x$$

$$-9A\cos 3x - 9B\sin 3x + 3A\sin 3x - 3B\cos 3x + A\cos 3x + B\sin 3x = 2\sin 3x$$

$$(-9A-3B+A)\cos 3x + (-9B+3A+B)\sin 3x = 2\sin 3x$$

$$(-8A-3B)\cos 3x + (-8B+3A)\sin 3x = 2\sin 3x$$

Undetermined Coefficients

Solution:

From the last equation

$$\boxed{(-8A-3B)}\cos 3x+\boxed{(-8B+3A)}\sin 3x=\boxed{0}\cos 3x+\boxed{2}\sin 3x$$

$$-8A-3B=0 \qquad\qquad -8B+3A=2$$

Solving gives, $A=\dfrac{6}{73}$ *and* $B=-\dfrac{16}{73}$

Thus, a particular solution $y_p=A\cos 3x+B\sin 3x$

$$y_p=\frac{6}{73}\cos 3x-\frac{16}{73}\sin 3x$$

Undetermined Coefficients

Example 3:

Find the general solution for $y''-2y'-3y=4x-5+6xe^{2x}$

Solution:

Step 1 *Find the solution for the associated homogeneous equation:*

$$y''-2y'-3y=0$$

The auxiliary equation $m^2-2m-3=0$

$b^2-4ac=(-2)^2-4(1)(-3)=16>0$ *(Case I)*

Find m_1 and m_2: $m^2-2m-3=(m+1)(m-3)=0$

Thus, $m_1=-1$ *and* $m_2=3$

For Case I, $y_c=c_1e^{m_1x}+c_2e^{m_2x}$

$$y_c=c_1e^{-x}+c_2e^{3x}$$

Undetermined Coefficients

Solution (continue...):

Step 2 *For DE* $y''-2y'-3y=4x-5+6xe^{2x}$

Notice that the nonhomogeneous part consists of 2 type of functions,

$$g(x)=\boxed{4x-5}+\boxed{6xe^{2x}}$$

Polynomial

Exponentials

In other words,

$$g(x)=4x-5+6xe^{2x}$$

$$g(x)=g_1(x)+g_2(x)$$

$$g(x)=polynomials+\exp onentials$$

Correspondingly, the superposition principle for the nonhomogeneous equations suggest that we can find the particular solution such that

$$y_p=y_{p_1}+y_{p_2}$$

where $y_{p_1}=Ax+B$ *and* $y_{p_2}=Cxe^{2x}+De^{2x}$

Undetermined Coefficients

Solution (continue...):

Step 2 (cont.)

Thus,

Differentiate twice,

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$
$$y_p' = A + Ce^{2x} + 2Cxe^{2x} + 2De^{2x}$$
$$y_p' = A + (C + 2D)e^{2x} + 2Cxe^{2x}$$
$$y_p'' = 2(C + 2D)e^{2x} + 2Ce^{2x} + 4Cxe^{2x}$$

$$y_p'' = 4(C + D)e^{2x} + 4Cxe^{2x}$$

Substitute y_p and its derivatives into DE

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$
$$-3Ax + (-2A - 3B) + (-3C)xe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}$$

which gives $-3A = 4$, $-2A - 3B = -5$, $-3C = 6$, $2C - 3D = 0$

Solve $A = -\dfrac{4}{3}, B = \dfrac{23}{9}, C = -2, D = -\dfrac{4}{3}$

Undetermined Coefficients

Solution (continue…):

Step 2 (cont.) *Particular solution*

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

Step 3 *The general solution*

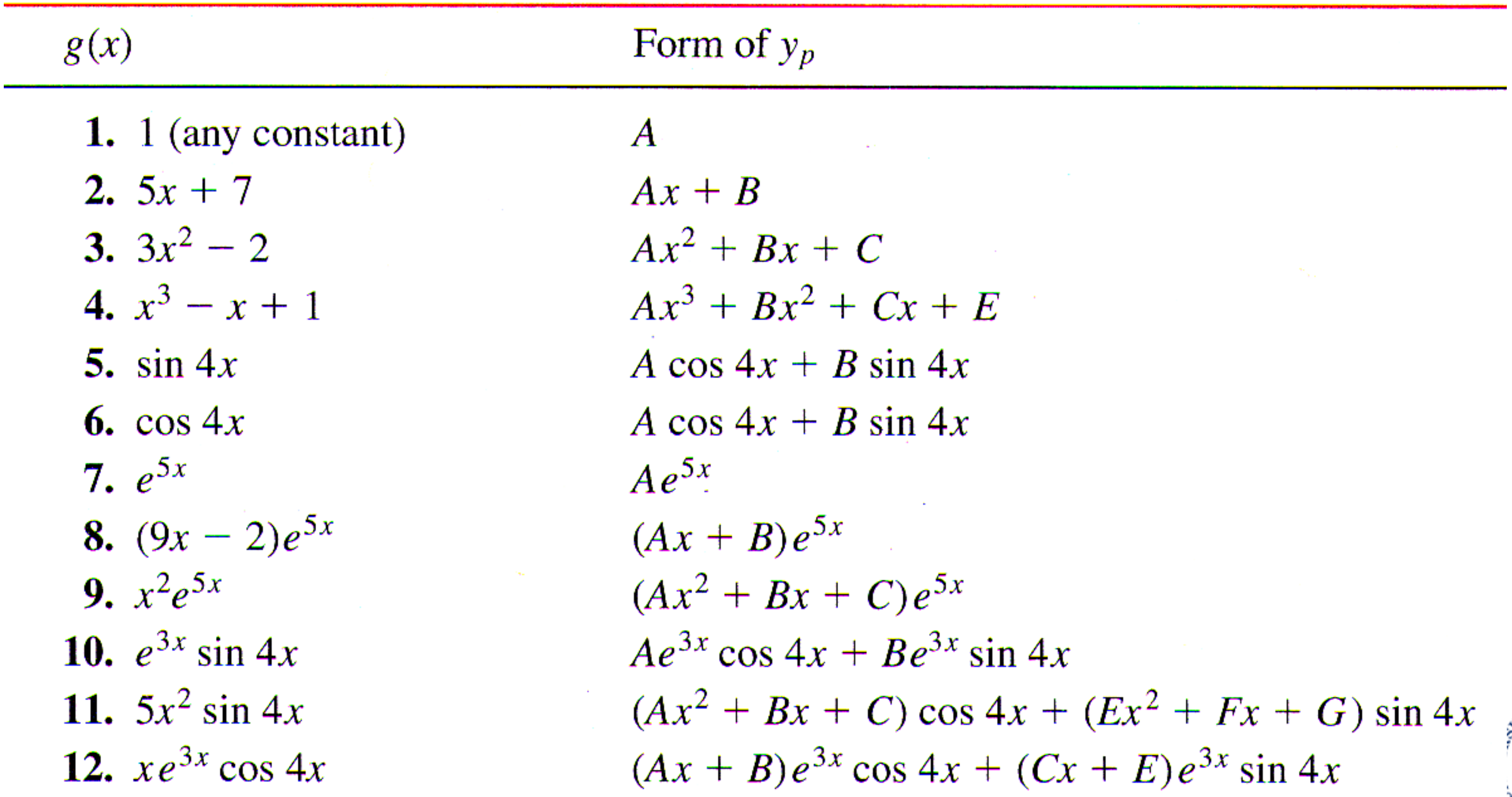
$$y = y_c + y_p$$

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

Undetermined Coefficients

Undetermined Coefficients – Superposition Approach

Some examples of g(x) and the corresponding form of the particular solutions:



$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Undetermined Coefficients

Example 4:

From the table in previous slide, determine the form of a particular solution y_p for:

(a) $y''-8y'+25y=5x^3e^{-x}-7e^{-x}$

(b) $y''+4y=x\cos x$

(c) $y''-9y'+14y=3x^2-5\sin 2x+7xe^{6x}$

(d) $y''-2y'+y=e^x$

Undetermined Coefficients

Solution:

(a) $y''-8y'+25y=5x^3e^{-x}-7e^{-x}$

Rewrite $g(x)=5x^3e^{-x}-7e^{-x}$

$$g(x)=(5x^3-7)e^{-x}$$

Refer to Item (9) in table, $y_p=(Ax^3+Bx^2+Cx+E)e^{-x}$
No duplication between y_p and y_c as y_c found to be $y_c=e^{4x}(c_1\cos 3x+c_2\sin 3x)$

(b) $y''+4y=x\cos x$

Function $g(x)=x\cos x$
Refer to Item (11) in table, $y_p=(Ax+B)\cos x+(Cx+E)\sin x$
No duplication between y_p and y_c as y_c found to be $y_c=c_1\cos 2x+c_2\sin 2x)$

Undetermined Coefficients

Solution:

(c) $y''-9y'+14y=3x^2-5\sin 2x+7xe^{6x}$

We know $y_p = y_{p_1}+y_{p_2}+y_{p_3}$

Thus, for

$$g(x)=3x^2 \qquad y_{p_1}=Ax^2+Bx+C$$

$$g(x)=-5\sin 2x \qquad y_{p_2}=E\cos 2x+F\sin 2x$$

$$g(x)=7xe^{6x} \qquad y_{p_3}=(Gx+H)e^{6x}$$

The assumption for the particular solution is then

$$y_p = y_{p_1}+y_{p_2}+y_{p_3}$$

$$y_p = Ax^2+Bx+C+E\cos 2x+F\sin 2x+(Gx+H)e^{6x}$$

No duplication between y_p and y_c as y_c found to be $y_c=c_1e^{2x}+c_2e^{7x}$

Undetermined Coefficients

Solution:

(d) $y''-2y'+y=e^x$

The complementary function $y_c = c_1e^x + c_2xe^x$

So, we cannot use the assumption that $y_p = Ae^x$ as this will duplicates y_c.

Also, we cannot use $y_p = Axe^x$ as again, this will duplicates y_c.

So, let's try $y_p = Ax^2e^x$

Differentiate y_p twice will gives
$$y_p' = 2Axe^x + Ax^2e^x$$
$$y_p'' = 2Ae^x + 4Axe^x Ax^2e^x$$

Substitute y_p and its derivatives into DE and solve gives $A = \frac{1}{2}$.

Thus the particular solution is $y_p = \frac{1}{2}x^2e^x$