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2^{nd} Order ODE: Homogeneous Linear Equations with Constant Coefficients

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Contemporary and Forward Looking

Second-Order Differential Equations

Learning Objective:

1. To solve the 2nd order ODE (Homogeneous Linear Equations) with Constant Coefficients.
2. To identify the types of 2nd order Ordinary Differential Equation

Second-Order Differential Equations

Chapter Contents:

- **Preliminary Theory on Linear Equations**
- **Reduction of Order**
- **Homogeneous Linear Equations with Constant Coefficients**
- Undetermined Coefficients
- Variation of Parameters
- Cauchy-Euler Equation

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

Recall back our first-order differential equation:

If you have a homogeneous linear equation of the first order

where coefficients $ay' + by = 0$ *and both a and b are constants;* $a \neq 0$

You know that you can solve it using separable variable method or linear equation method.
BUT, do you know that you can also solve the above DE using simple algebra.

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

Let us try this,

$$ay' + by = 0$$

$$y' = -\frac{b}{a}y$$

Let, $k = -\frac{b}{a}$:

$$y' = ky$$ where k is a constant

Note that the only basic function whose derivative is a constant multiple of itself is an exponential function e^{mx}.

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

So, the new solution method:

Substitute $y = e^{mx}$ and its derivative $y' = me^{mx}$ into

$$ay' + by = 0$$

This gives

$$ame^{mx} + be^{mx} = 0$$

$$e^{mx}(am + b) = 0$$

Since e^{mx} is never zero for real values of x, the above equation can only be satisfied if $am + b = 0$.
By solving the value of m, we have obtained $y = e^{mx}$ which is a solution of the first order differential equation.

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

Let's apply this method of solution to a second-order differential equation

$$ay''+by'+cy=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

where a, b and c are constants.

If we try to find a solution of the form $y=e^{mx}$, substitute $y=e^{mx}$ and its derivatives $y'=me^{mx}$ and $y''=m^2e^{mx}$ into (1) gives

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$
$$e^{mx}(am^2+bm+c)=0$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

Again, as e^{mx} is never zero for all real value of x

$$am^2 + bm + c = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_} (2)$$

Equation (2) above is called the **auxiliary equation**.

When ***m*** is chosen as a root of the quadratic equation (2), the two roots will be

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Or, $m_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $m_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

Thus, based on the two roots, there will be **3 forms of the general solution** of equation $ay''+by'+cy=0$ corresponding to the 3 cases:

Case I: m_1 **and** m_2 **are real and distinct** $(b^2-4ac>0)$,

Case II: m_1 **and** m_2 **are real and equal** $(b^2-4ac=0)$,

Case III: m_1 **and** m_2 **are conjugate complex number** $(b^2-4ac<0)$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

CASE I: DISTINCT REAL ROOTS $(b^2 - 4ac > 0)$

Distinct means different or unequal.
Under the assumption that m_1 and m_2 are unequal, we can find the 2 solutions,

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}$$

Verified that y_1 and y_2 are linearly independent , thus form a fundamental set of solutions.

Thus, the general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

CASE II: REPEATED REAL ROOTS $(b^2 - 4ac = 0)$

When $m_1 = m_2$, we obtain only ONE solution $y_1 = e^{m_1 x}$.

We can find the second solution from reduction of order formula:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^{\;2}} dx$$

From $$ay'' + by' + cy = 0$$

Rewrite $$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

CASE II: REPEATED REAL ROOTS $(b^2 - 4ac = 0)$

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

Find $e^{-\int P(x)dx} = e^{-\int \frac{b}{a} dx}$

From quadratic equation, $m_1 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

when $b^2 - 4ac = 0$, $m_1 = -\dfrac{b}{2a}$ **or** $-\dfrac{b}{a} = 2m_1$

Thus, integrating factor,

$$e^{-\int \frac{b}{a} dx} = e^{\int 2m_1 dx} = e^{2m_1 x}$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

CASE II: REPEATED REAL ROOTS $(b^2 - 4ac = 0)$

Using the formula:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^{\,2}} dx$$

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx$$

$$y_2 = e^{m_1 x} \int dx$$

$$y_2 = x e^{m_1 x}$$

Thus, the general solution is $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

CASE III: COMPLEX ROOTS $(b^2 - 4ac < 0)$

If m_1 and m_2 are complex, then we can write

$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

where α and β are real and $i = \sqrt{-1}$.

From
$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

We obtain
$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

Or,
$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'-25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(a) $2y''-5y'-3y=0$

Let, a = 2, b = -5 and c = -3 and auxiliary equations is $2m^2 - 5m - 3 = 0$.

$$b^2-4ac=(-5)^2-4(2)(-3)=49>0$$

This is Case I where m_1 and m_2 are distinct real roots $(b^2-4ac>0)$.

Thus,

$$m_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad m_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$m_1=3 \qquad m_2=-\frac{1}{2}$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'-25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(a) $2y''-5y'-3y=0$

For Case I, general solution is $y=c_1e^{m_1x}+c_2e^{m_2x}$

Thus, $y=c_1e^{3x}+c_2e^{-\frac{1}{2}x}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'+25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(b) $y''-10y'+25y=0$

Let, a = 1, b = -10 and c = 25 and auxiliary equations is $m^2 - 10m + 25 = 0$.

$$b^2-4ac=(-10)^2-4(1)(25)=0$$

This is Case II where m_1 and m_2 are repeated real roots $(b^2-4ac=0)$.
Thus, $\mathbf{m_1 = m_2}$.

$$m^2-10m+25=(m-5)(m-5)=0$$

Thus, $\mathbf{m_1 = m_2 = 5}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'+25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(b) $y''-10y'+25y=0$

For Case II, general solution is $y=c_1e^{m_1x}+c_2xe^{m_1x}$

Thus, $y=c_1e^{5x}+c_2xe^{5x}$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'+25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(c) $y''+4y'+7y=0$

Let, a = 1, b = 4 and c = 7 and auxiliary equations is $m^2+4m+7=0$.

$$b^2-4ac=(4)^2-4(1)(7)=-12<0$$

This is Case III where m_1 and m_2 are complex roots $(b^2-4ac<0)$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'+25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(c) $y''+4y'+7y=0$

From $m^2+4m+7=0$:

$$m_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad m_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$m_1=\frac{-4+\sqrt{-12}}{2} \qquad m_2=\frac{-4-\sqrt{-12}}{2}$$

$$m_1=\frac{-4+2\sqrt{3}i}{2} \qquad m_2=\frac{-4-2\sqrt{3}i}{2}$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Examples:

Solve the following differential equations.

(a) $2y''-5y'-3y=0$ (b) $y''-10y'+25y=0$ (c) $y''+4y'+7y=0$

Solutions:

(c) $y''+4y'+7y=0$

$m_1 = -2+\sqrt{3}i$ $m_2 = -2-\sqrt{3}i$

which gives $\alpha = -2$ and $\beta = \sqrt{3}$

For Case III, general solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

Thus,

$$y = c_1 e^{-2x}\cos\sqrt{3}x + c_2 e^{-2x}\sin\sqrt{3}x$$

2nd-Order DE: Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Equations with Constant Coefficients:

			General Solution
Case I	m_1 and m_2 are real $m_1 \neq m_2$	$b^2 - 4ac > 0$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Case II	m_1 and m_2 are real $m_1 = m_2$	$b^2 - 4ac = 0$	$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$
Case III	m_1 and m_2 are conjugate complex number	$b^2 - 4ac < 0$	$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$ $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$