# 2nd Order ODE: Reduction of Order

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# Second-Order Differential Equations

Learning Objective:

1. To apply reduction of order to Homogeneous 2nd order ODE.
2. To identify the types of 2nd order Ordinary Differential Equation

# Second-Order Differential Equations

**Chapter Contents:**

- **Preliminary Theory on Linear Equations**
- **Reduction of Order**
- Homogeneous Linear Equations with Constant Coefficients
- Undetermined Coefficients
- Variation of Parameters
- Cauchy-Euler Equation

# Reduction of Order

# 2nd-Order DE: Reduction of Order

## Reduction of Order:

Given, a homogeneous second-order differential equation

$$y''+p(x)y'+q(x)y=0 \qquad \text{———————— (1)}$$

We want to find TWO independent solutions.

Reduction of order is a technique for **finding a second solution**, if the first solution is known.

# 2nd-Order DE: Reduction of Order

## Reduction of Order:

$$y''+p(x)y'+q(x)y=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

The basic idea is that equation (1) above can be reduced to a linear first-order DE by means of a substitution involving the known solution $y_1$.

A second solution $y_2$ can be found after the first-order DE is solved.

# 2nd-Order DE: Reduction of Order

## Reduction of Order:

$$y''+p(x)y'+q(x)y=0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (1)}$$

Suppose we know $y_1$ is a solution of equation (1). Now we want to find $y_2$.

For $y_1$ and $y_2$ to form a set of fundamental solutions, $y_1$ and $y_2$ must be linearly independent, i.e.

$$\frac{y_2(x)}{y_1(x)} \neq \text{constant}$$

# 2nd-Order DE: Reduction of Order

## Reduction of Order:

So, let $$\frac{y_2(x)}{y_1(x)} = u(x)$$

Or, $$y_2(x) = y_1(x).u(x)$$

The function u(x) can be found by substituting

$$y_2(x) = y_1(x).u(x)$$

into the given differential equation.

# 2nd-Order DE: Reduction of Order

**Example 1**:
Given that $y_1 = e^x$ is a solution of $y''-y=0$.
Use reduction of order to find a second solution $y_2$.

**Solution**:

By reduction of order, let
$$y_2 = u.y_1$$
$$y_2 = ue^x$$

Differentiate $y_2$ twice (by Product Rule):
$$y_2' = u'e^x + ue^x$$
$$y_2'' = ue^x + 2e^x u' + e^x u''$$

For $y_2$ to be a solution of the differential equation $y''-y=0$:
$$(y_2)''-y_2 = 0$$

# 2nd-Order DE: Reduction of Order

**Solution**:

From $$(y_2)''-y_2=0$$

Substitute $$\underbrace{ue^x+2e^x u'+e^x u''}_{y_2''}-\underbrace{ue^x}_{y_2}=0$$

Rewrite, $$e^x(2u'+u'')=0$$

Since $y_1=e^x$ is a solution, $e^x\neq 0$.

Thus, we need $2u'+u''=0$.

# 2nd-Order DE: Reduction of Order

**Solution**:

To solve $2u' + u'' = 0$ , we use substitution.

Let $w = u'$ :

$$2u' + u'' = 0$$
$$2w + w' = 0$$
$$w' + 2w = 0$$

Which reduced to a linear first-order equation.

By linear equation method of solving, we found the integrating factor, $e^{2x}$:

And
$$\frac{d}{dx}\left[e^{2x} w\right] = 0$$

Integrate
$$w = c_1 e^{-2x}$$

# 2nd-Order DE: Reduction of Order

**Solution**:

From $w = c_1 e^{-2x}$

Substitute back, $u' = c_1 e^{-2x}$

Integrates again, $u = \int c_1 e^{-2x} dx$

$$u = -\frac{1}{2} c_1 e^{-2x} + c_2$$

From $y_2 = u.y_1$:

$$y_2 = \left( -\frac{1}{2} c_1 e^{-2x} + c_2 \right) e^{x}$$

$$y_2 = -\frac{1}{2} c_1 e^{-x} + c_2 e^{x}$$

# 2nd-Order DE: Reduction of Order

**Solution:**

From $$y_2 = -\frac{1}{2}c_1e^{-x} + c_2e^{x}$$

By choosing $c_2 = 0$ and $c_1 = -2$, $y_2 = e^{-x}$ .

Check that $y_1$ and $y_2$ are linearly independent using Wronskian:

$$W(e^{x}, e^{-x}) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2 \neq 0$$

Thus, shown that $y_1 = e^{x}$ and $y_2 = e^{-x}$ are linearly independent solutions of a linear second-order equation, thus form a general solution

$$y = c_1e^{x} + c_2e^{-x}$$

# 2nd-Order DE: Reduction of Order

**Reduction of Order (by Formula):**

Second solution $y_2$ for $y'' + p(x)y' + q(x)y = 0$

can also be obtained using the formula,

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^{\,2}} dx$$

# 2nd-Order DE: Reduction of Order

**Example 2 (Finding second solution by formula):**

Given that $y_1 = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$.
Find the general solution of the differential equation.

**Solution:**

From $$x^2 y'' - 3xy' + 4y = 0$$

Rewrite to standard form $$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$$

Using the formula $$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^{\,2}} dx$$ ------------------------ (1)

where $y_1 = x^2$ and $e^{-\int P(x)dx} = e^{3\int \frac{1}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

# 2nd-Order DE: Reduction of Order

**Example 2 (Finding second solution by formula):**

Given that $y_1 = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$.
Find the general solution of the differential equation.

**Solution:**

Substitute into formula (1): $y_2 = x^2 \int \frac{x^3}{x^4} dx$

$$y_2 = x^2 \int \frac{1}{x} dx$$

$$y_2 = x^2 \ln x$$ *(Ignore the constant after integration)*

Thus, the general solution $y = c_1 y_1 + c_2 y_2$ is

$$y = c_1 x^2 + c_2 x^2 \ln x$$

# 2nd-Order DE: Reduction of Order

**Sample Exam Question:**

Function $y_1 = e^{x/3}$ is a solution of $6y'' + y' - y = 0$.
Find the second solution $y_2$.

**Solution:**

*By reduction of order:*

Let second solution, $y_2 = ue^{\frac{1}{3}x}$

Differentiate $y_2$ twice:

$$y_2' = u'e^{\frac{1}{3}x} + \frac{1}{3}ue^{\frac{1}{3}x}$$

$$y_2'' = u''e^{\frac{1}{3}x} + \frac{2}{3}u'e^{\frac{1}{3}x} + \frac{1}{9}ue^{\frac{1}{3}x}$$

# 2nd-Order DE: Reduction of Order

## Sample Exam Question:

Function $y_1 = e^{x/3}$ is a solution of $6y'' + y' - y = 0$.
Find the second solution $y_2$.

## Solution (continue):

Substitute $y_2$ and its derivatives into (1):

$$6u''e^{\frac{1}{3}x} + 4u'e^{\frac{1}{3}x} + \frac{2}{3}ue^{\frac{1}{3}x} + u'e^{\frac{1}{3}x} + \frac{1}{3}ue^{\frac{1}{3}x} - ue^{\frac{1}{3}x} = 0$$

$$6u''e^{\frac{1}{3}x} + 5u'e^{\frac{1}{3}x} = 0$$

$$6u'' + 5u' = 0$$

$$u'' + \frac{5}{6}u' = 0$$

# 2nd-Order DE: Reduction of Order

**Sample Exam Question**

Function $y_1 = e^{x/3}$ is a solution of $6y'' + y' - y = 0$.
Find the second solution $y_2$.

**Solution (continue):**

Let w = u': $w' + \frac{5}{6}w = 0$

By linear equation, integrating factor is $e^{\frac{5}{6}x}$ :

$$\int \frac{d}{dx}\left[e^{\frac{5}{6}x}.w\right] = \int 0\,dx$$

$$e^{\frac{5}{6}x}.w = c \quad \text{and} \quad w = ce^{-\frac{5}{6}x}$$

# 2nd-Order DE: Reduction of Order

**Sample Exam Question:**

Function $y_1 = e^{x/3}$ is a solution of $6y'' + y' - y = 0$.
Find the second solution $y_2$.

**Solution (continue):**

Substitute back, $u' = ce^{-\frac{5}{6}x}$ and integrate gives $u = c_1 e^{-\frac{5}{6}x}$

Thus, second solution, $y_2 = e^{-\frac{5}{6}x}(e^{\frac{1}{3}x}) = e^{\frac{-1}{2}x}$