2nd Order ODE

Hishammudin Afifi Bin Huspi
Faculty of Engineering
Universiti Malaysia Sarawak

Second-Order Differential Equations

Learning Objective:

1. To understand the preliminary theory on linear equations
2. To identify the types of 2nd order Ordinary Differential Equation
3. To apply the 2nd order Ordinary Differential Equation

Second-Order Differential Equations

We can solve first-order DE by recognizing them as:

- Separable
- Linear equations
- Exact equations
- Solutions By Substitutions

Second-Order Differential Equations

- Ordinary differential equations (ODEs) can be divided into two larger classes, linear ODEs and non linear ODEs.

- Non linear 2^{nd} order ODEs are generally difficult to solve, whereas linear 2^{nd} order ODE are much simpler because there are standard methods for solving many of these equations.

- In the following chapters, our main goal will be to find solution for <u>second order linear</u> differential equation.

Second-Order Differential Equations

Chapter Contents:

- Preliminary Theory on Linear Equations
- Reduction of Order
- Homogeneous Linear Equations with Constant Coefficients
- Undetermined Coefficients
- Variation of Parameters
- Cauchy-Euler Equation

Preliminary Theory on Linear Equations

A linear second-order differential equation has the form :

$$R(x)y''+P(x)y'+Q(x)y=S(x)$$

When R(x) ≠ 0, we can divide this equation by R(x) and obtain the special linear equation:

$$y''+p(x)y'+q(x)y=f(x) \qquad \text{\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

So, how can we produce all solutions of equation (1)?

Preliminary Theory on Linear Equations

Theory of Solutions of *y'' + p(x)y' + q(x)y = f(x)*

Consider a simple 2nd order linear equation as below:

$$y''-12x=0$$

We can also write as $y''=12x$

Integrate gives $y'=\int y''\,dx=\int 12x\,dx=6x^2+C$

Integrate again $y=\int y'\,dx=\int (6x^2+c)\,dx=2x^3+Cx+K$

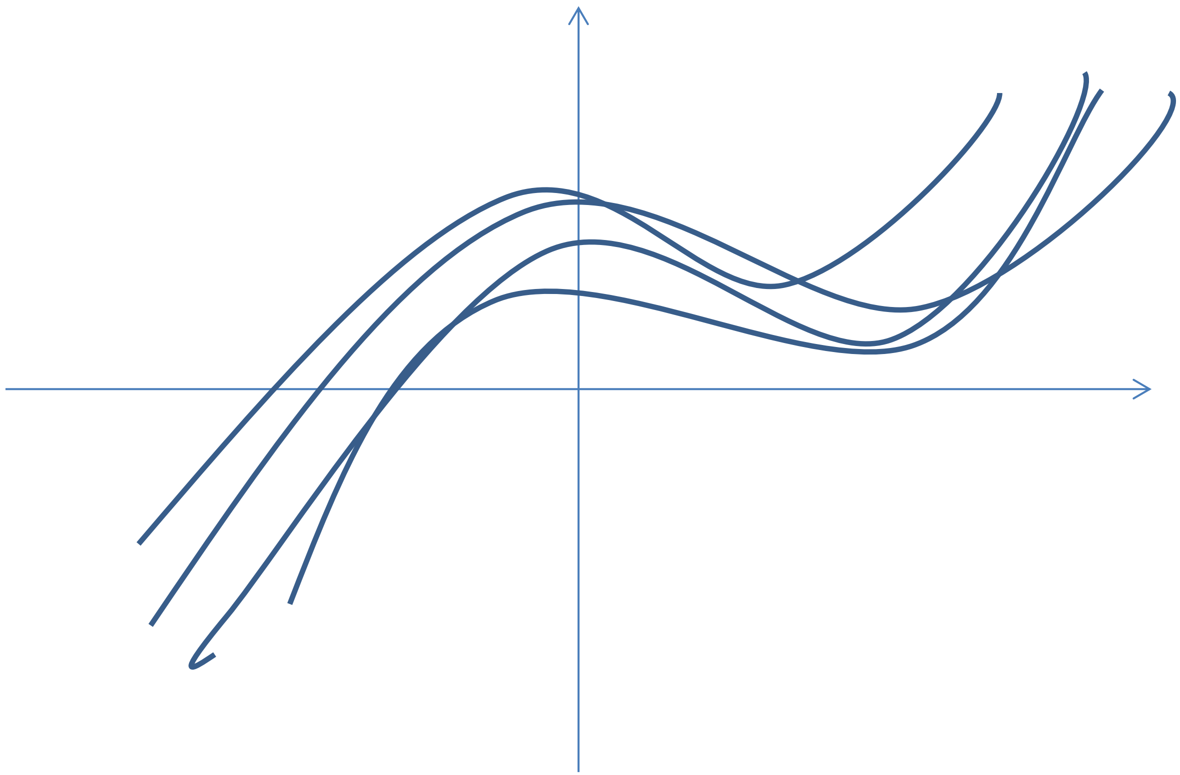
This solution is defined for all value of x, and contains TWO arbitrary constants. It is natural that the solution of a second order equation, involving TWO integrations, should contain **TWO arbitrary constants**.

Preliminary Theory on Linear Equations

Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$

For different values of constants C and K, the integral curves for

$$y = 2x^3 + Cx + K$$



Preliminary Theory on Linear Equations

Theory of Solutions of *y'' + p(x)y' + q(x)y = f(x)*

Suppose we want a solution satisfying the initial condition:

$$y = 2x^3 + Cx + K \qquad y(0) = 3$$

Then, $y = 2(0) + c(0) + K = 3$

$$K = 3$$

Thus, the general solution $y = 2x^3 + Cx + 3$

Where all the solutions will pass through (0,3). Some of these curves are as shown in following slide.

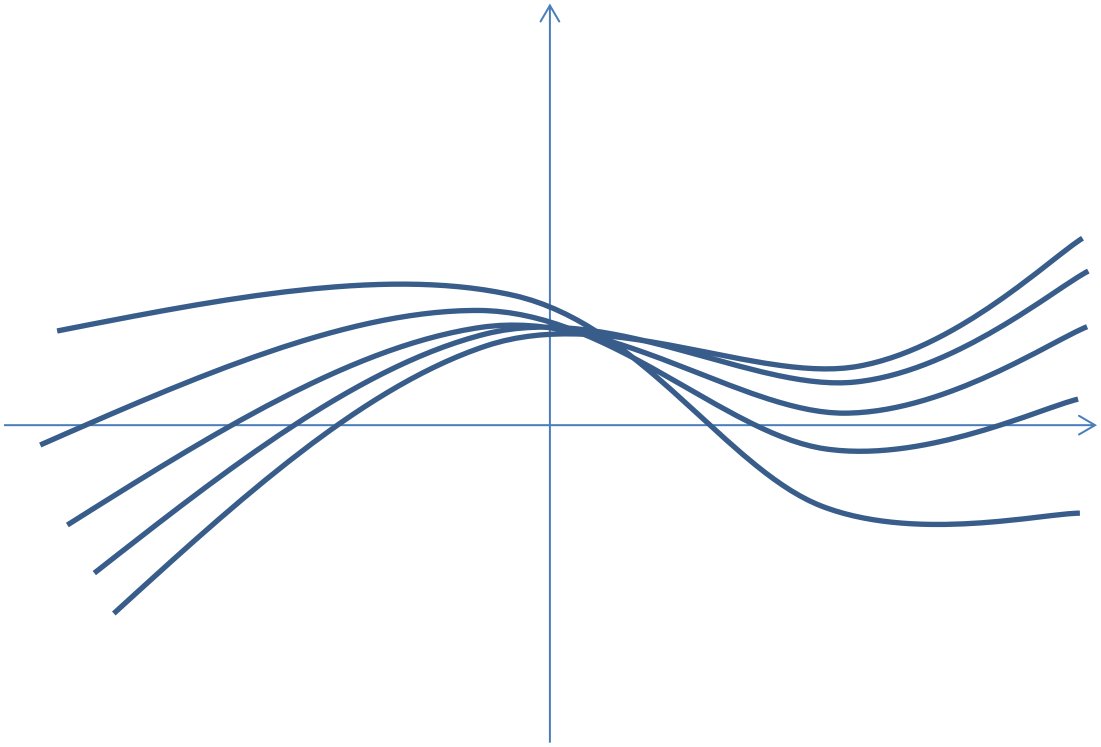
Preliminary Theory on Linear Equations

Theory of Solutions of *y'' + p(x)y' + q(x)y = f(x)*

For different values of constants C, the integral curves for

$$y = 2x^3 + Cx + 3$$

where all solutions pass through (0,3)



Preliminary Theory on Linear Equations

Theory of Solutions of *y″ + p(x)y′ + q(x)y = f(x)*

Suppose, we also specify the initial condition:

$$y = 2x^3 + Cx + K \qquad y'(0) = -1$$

From

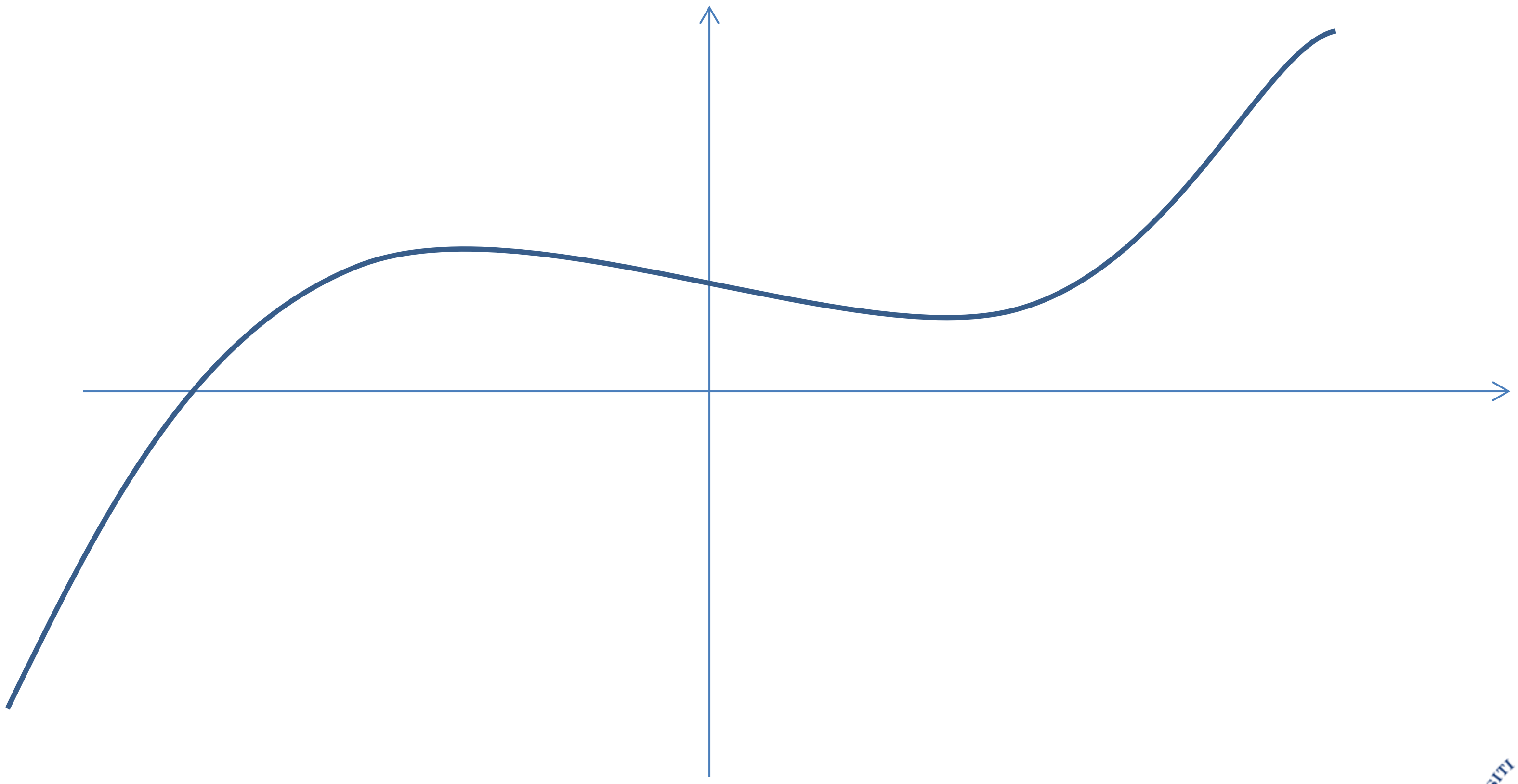
$$y' = 6x^2 + C$$
$$-1 = 6(0)^2 + C$$
$$C = -1$$

Thus, the exact solution that satisfies both initial conditions $y(0) = 3$ and $y'(0) = -1$ is

$$y(x) = 2x^3 - x + 3$$

Preliminary Theory on Linear Equations

Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$



Preliminary Theory on Linear Equations

Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$

From this example, we can summarize that:

1. General solution for a second-order differential equation involved TWO arbitrary constants.

2. Initial condition $y(0) = 3$, specifies that the solution curve must pass through (0,3) and determined one of the constants.

3. Initial condition $y'(0) = -1$, selects the solution curve that pass through (0,3) with the slope of -1 and gives a unique solution to the problem.

Preliminary Theory on Linear Equations

Theory of Solutions of $y'' + p(x)y' + q(x)y = f(x)$

Thus, from $$y'' + p(x)y' + q(x)y = f(x)$$

There are TWO initial conditions:
One specifying **a point** lying on the solution curve, and the other **its slope** at that point.

This problem has the form

$$y'' + p(x)y' + q(x)y = f(x); \quad y(x_0) = A, \quad y'(x_0) = B$$

in which A and B are given real numbers.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

When f(x) is identically zero in $y''+p(x)y'+q(x)y = f(x)$

$$y''+p(x)y'+q(x)y=0$$

is called **homogenous**.

Homogeneous simply means the right side of the equation is ZERO.

Example:

$2y''+3y'-5y=0$	Homogeneous linear 2nd order DE
$xy''+y'+xy=0$	Homogeneous linear 2nd order DE
$y''+25y=e^{-x}\cos x$	Nonhomogeneous linear 2nd order DE

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs: Superposition Principle

For the homogeneous equation, the backbone of this structure is the superposition principle or linearity principle, which says that:

Further solutions can be obtain from given ones by adding them or by multiplying them with any constants.

This is a great advantage of homogeneous linear ODEs.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs: Superposition Principle

Example 1:

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0 \quad \text{for all } x.$$

Let's verify for $y = \cos x$; $y' = -\sin x$; $y'' = -\cos x$

Substitute in $y'' + y = (-\cos x) + \cos x = 0$ (*verified*)

Verify for $y = \sin x$; $y' = \cos x$; $y'' = -\sin x$

Substitute in $y'' + y = (-\sin x) + \sin x = 0$ (*verified*)

If we multiply $\cos x$ by any constant, say 4.7 and $\sin x$ by, say -2:

$$(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x) = -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x = 0$$

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs: Superposition Principle

From Example 1, we have obtained $y_1 = \cos x$ and $y_2 = \sin x$, and this gives a function of the form:

$$y = c_1 y_1 + c_2 y_2$$ (c_1 and c_2 are arbitrary constants)

This is called a **linear combination** of y_1 and y_2.

From this, we can formulate the result by **superposition principle** or **linearity principle**.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs: Superposition Principle

Theorem on Superposition Principle – Homogeneous Equations

Let y_1 and y_2 be solutions of the homogeneous linear second order differential equation on an interval I

$$y'' + p(x)y' + q(x)y = 0$$

Then the linear combination $y = c_1 y_1 + c_2 y_2$

is also a solution on the interval. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Note: This theorem does not apply for Nonhomogeneous equations.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

The point of taking linear combinations $c_1 y_1 + c_2 y_2$ is to obtain more solutions from just two solutions.

However, if y_2 is already a constant multiple of y_1, then

$$c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 k y_1 = (c_1 + k c_2) y_1$$

is just another constant multiple of y_1.
Thus y_2 is redundant.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

Linear Dependence or Linear Independence

Two functions f and g are said to be **linearly dependent** on an interval I if, for some constant c, either $f(x)=cg(x)$ for all x in I, or $g(x)=cf(x)$ for all x in I.

If f and g are not linearly dependent on I, then they are said to be **linearly independent** on the interval.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

Simple test to determine whether TWO solutions are linearly independent on an interval:

Wronskian

Suppose each solutions y_1 and y_2 possess at least n-1 derivatives. The 2x2 determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is called the Wronskian of the functions.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

For linearly independent solutions

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0$$

for every x in the interval.

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

This gives further definition of solutions:

1. If y_1 and y_2 are linearly independent, these are called **Fundamental Set of Solutions**.

2. If y_1 and y_2 are Fundamental Set of Solutions, then the **general solution** of the equation is

$$y = c_1 y_1 + c_2 y_2$$

Preliminary Theory on Linear Equations

Homogeneous Linear 2nd Order ODEs

Example 2:
The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval (-∞,∞).
Verify by inspection that the solutions are linearly independent on the x-axis.

Solution:

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{3x} \end{vmatrix} = -6 \neq 0$$ Linearly independent

Thus, y_1 and y_2 form a fundamental set of solutions, and consequently,

$y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation.

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

Similar ideas used for homogeneous equation can be used to develop solution for nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

Let y_1 and y_2 be a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Let y_p be any solution of equation $y'' + p(x)y' + q(x)y = f(x)$

Then, the solution Φ will be $\varphi = c_1 y_1 + c_2 y_2 + y_p$

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

Thus, for equation $y''+p(x)y'+q(x)y = f(x)$ the **general solution** is $\varphi = c_1 y_1 + c_2 y_2 + y_p$

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

In summary, we can solve nonhomogeneous equation $y''+p(x)y'+q(x)y = f(x)$ by the following strategy:

1. Find the general solution $c_1y_1 + c_2y_2$ of the associated homogeneous equation $y''+p(x)y'+q(x)y = 0$
2. Find any solution y_p of $y''+p(x)y'+q(x)y = f(x)$
3. Write the general solution $c_1y_1 + c_2y_2 + y_p$ This expression contains all possible solutions of the nonhomogeneous differential equation.

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

In other words, the general solution of the nonhomogeneous equation is then

y = complementary functions + any particular functions

$$y = y_c + y_p$$

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

Example 3:

Verify that the given function is the general solution of the nonhomogeneous differential equation:

$$y''-7y'+10y=24e^{x}$$
$$y=c_1e^{2x}+c_2e^{5x}+6e^{x}$$

Solution 3:

Nonhomogeneous equation, $y''-7y'+10y=24e^{x}$ -------------- (1)

Associated homogeneous equations, $y''-7y'+10y=0$ --------- (2)

If function $y=c_1e^{2x}+c_2e^{5x}+6e^{x}$ is a general solution to equation (1), then function $y=c_1e^{2x}+c_2e^{5x}$ is a general solution to equation (2).

Let $y=c_1e^{2x}+c_2e^{5x}$ ------------------------------ (3)

$$y=c_1y_1+c_2y_2$$

First, prove that both y_1 and y_2 are both solutions to DE (2).
Differentiate y_1 and y_2 twice :

$$y_1=e^{2x} \qquad y_2=e^{5x}$$

$$y_1'=2e^{2x} \qquad y_2'=5e^{5x}$$

$$y_1''=4e^{2x} \qquad y_2''=25e^{5x}$$

Solution 3 (continue):

Substitute y_1 and its derivatives into left-hand side of DE (2):

LHS: $$4e^{2x} - 7(2e^{2x}) + 10e^{2x} = 0$$

RHS = 0, thus verified that y_1 is a solution of the DE.

Substitute y_2 and its derivatives into left-hand side of DE (2):

LHS: $$25e^{5x} - 7(5e^{5x}) + 10e^{5x} = 0$$

RHS = 0, thus verified that y_2 is a solution of the DE.

Definition of a General Solution:

1. *If y_1 and y_2 are linearly independent, these are called Fundamental Set of Solutions.*
2. *If y_1 and y_2 are Fundamental Set of Solutions, then the general solution of the equation is*

$$y = c_1 y_1 + c_2 y_2$$

So next, prove that y_1 and y_2 form a Fundamental Set of Solutions by proving that they are linearly independent.

Solution 3 (continue):

Using Wronskian method:

$$W(e^{2x}, e^{5x}) = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix}$$

$$W(e^{2x}, e^{5x}) = e^{2x}.5e^{5x} - 2e^{2x}.e^{5x}$$

$$W(e^{2x}, e^{5x}) = 3e^{7x} \neq 0$$

Verified that both y_1 and y_2 are:

1. Solutions for the homogenous linear DE $y''-7y'+10y=0$
2. Linearly independent.

Thus, y_1 and y_2 form a Fundamental Set of Solution.

Next, prove that y_p is the solution for the nonhomogeneous equation (1).

Solution 3 (continue):

If function $y = c_1 e^{2x} + c_2 e^{5x} + 6e^{x}$ is a valid general solution, then

$$y = c_1 y_1 + c_2 y_2 + y_p$$

Thus, $y_p = 6e^{x}$

To substitute y_p into nonhomogeneous equation (1), differentiate twice:

$$y_p = 6e^{x}$$

$$y_p' = 6e^{x}$$

$$y_p'' = 6e^{x}$$

Solution 3 (continue):

Substitute y_p and its derivative into left-hand side of (1):

LHS: $$6e^x - 7(6e^x) + 10(6e^x) = 6e^x - 42e^x + 60e^x$$
$$= 24e^x$$

RHS $= 24e^x$

Found, LHS = RHS, thus verified that $y_p = 6e^x$ is a solution for the nonhomogeneous equation.

Preliminary Theory on Linear Equations

Nonhomogeneous Linear 2nd Order ODEs

In summary, to prove that a function is a general solution for a nonhomogeneous equation:

1. Prove that y_1 and y_2 are both solutions for the associated homogeneous equation.
2. If both y_1 and y_2 are the solutions, prove that both form a fundamental set of solutions by checking if they are linearly independent (using Wronskian).
3. If both y_1 and y_2 are the fundamental set of solutions, then $c_1y_1 + c_2 y_2$ form the general solution for the homogeneous part.
4. Prove that yp is a particular solution for the nonhomogeneous equation.